UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH7202**

ASSESSMENT PATTERN : **MATH7202A**

MODULE NAME : **Algebra 4: Groups and Rings**

DATE : **14-May-10**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2009/10-MATH7202A-001-EXAM-175

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Describe the group structure on the set $\mathrm{Aut}(K)$ of automorphisms of a group K.

   If $\varphi : Q \to \mathrm{Aut}(K)$ is a group homomorphism, explain what is meant by the *semi-direct product* $K \rtimes_\varphi Q$ and show it is a group.

   Describe explicitly

   (i) the group structure on $\mathrm{Aut}(C_{14})$; (ii) all homomorphisms $\varphi : C_3 \to \mathrm{Aut}(C_{14})$.

   How many isomorphically distinct groups are there of the form $C_{14} \rtimes_\varphi C_3$? Explain.

2. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X, and let $x \in X$. Explain what is meant by

   (i) the orbit, $\langle x \rangle$, of x ; (ii) the *stability group* G_x of x .

   Explain, with proof, what is meant by the *class equation* of the action in both its set-theoretic and numerical forms.

   Let $G = D_{14}$, the dihedral group of order 14 and let G act on itself by *conjugation*

   $$\circ \; : \; D_{14} \times D_{14} \to D_{14} \; ; \; g \circ h = ghg^{-1}.$$

   Give an explicit description of (i) the orbits in this action ;

   (ii) the stability subgroup of a representative element in each orbit ;

   (iii) both forms of the class equation.

3. Let P, Q be subgroups of a group G; explain what is meant by saying that P *normalizes* Q. Show that, when P normalizes Q, there is a group isomorphism

   $$PQ/Q \;\cong\; P/(P \cap Q).$$

   State and prove Wilson's Theorem.

   Let p be a prime, and let G be a group of order kp^n where $n \geqslant 1$ and k is coprime to p, and let N_p be the number of subgroups of G of order p^n. Assuming that $N_p \geqslant 1$, show that $N_p \equiv 1 \pmod p$.

4. Let K, Q be subgroups of coprime order in finite group G and suppose also that $K \lhd G$ and $|G| = |K||Q|$. Prove that G is a semi-direct product $G \cong K \rtimes Q$.

   Let G be a group of order 725 ; show that G has a normal subgroup of order 29. Hence classify all groups of order 725 up to isomorphism, giving a set of generators and relations for each isomorphism type.

   (You may use Sylow's Theorem without proof provided it is correctly stated.)

5. Let A be a commutative integral domain which contains a field $\mathbb{F}$ as a subring and is such that $\dim_{\mathbb{F}}(A)$ is finite. Show that A is a field.

   Show that both $x^2 + x + 2$ and $x^2 + 2x + 3$ are irreducible over the field $\mathbb{F}_5$.

   Find an explicit ring isomorphism

   $$\varphi : \mathbb{F}_5[x]/(x^2 + x + 2) \to \mathbb{F}_5[x]/(x^2 + 2x + 3).$$

   Show that $x^6 = 3$ in $\mathbb{F}_5[x]/(x^2 + 2x + 3)$ and thereby deduce that x generates the unit group $[\mathbb{F}_5[x]/(x^2 + 2x + 3)]^*$.

   Hence or otherwise, find a generator for $[\mathbb{F}_5[x]/(x^2 + x + 2)]^*$.

6. State Gauss' Lemma.

   Let $a(x) \in \mathbb{Z}[x]$ be a integral polynomial of degree > 0. Explain what is meant by saying that $a(x)$ has a *proper factorization* over $\mathbb{Z}$.

   Prove that if $a(x)$ has no proper factorization over $\mathbb{Z}$ then it is irreducible over $\mathbb{Q}$.

   In each case below, decide, justifying your decision, whether or not the given polynomial is irreducible over $\mathbb{Q}$;

   (i) $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 12$;

   (ii) $x^{12} + 2x^6 + 1$.

   If the polynomial is not irreducible, give its complete factorization into $\mathbb{Q}$-irreducible factors.